ON GRAVITATIONAL WAVES.

BY

A. EINSTEIN and N. ROSEN.

ABSTRACT.

The rigorous solution for cylindrical gravitational waves is given. For the convenience of the reader the theory of gravitational waves and their production, already known in principle, is given in the first part of this paper. After encountering relationships which cast doubt on the existence of *rigorous* solutions for undulatory gravitational fields, we investigate rigorously the case of cylindrical gravitational waves. It turns out that rigorous solutions exist and that the problem reduces to the usual cylindrical waves in euclidean space.

I. APPROXIMATE SOLUTION OF THE PROBLEM OF PLANE WAVES AND THE PRODUCTION OF GRAVITATIONAL WAVES.

It is well known that the approximate method of integration of the gravitational equations of the general relativity theory leads to the existence of gravitational waves. The method used is as follows: We start with the equations

$$R_{\mu\nu} - \tfrac{1}{2}g_{\mu\nu}R = - T_{\mu\nu}. \tag{1}$$

We consider that the $g_{\mu\nu}$ are replaced by the expressions

$$g_{\mu\nu} = \delta_{\mu\nu} + \gamma_{\mu\nu}, \tag{2}$$

where

$$\delta_{\mu\nu} = 1 \quad \text{if} \quad \mu = \nu,$$
$$\quad = 0 \quad \text{if} \quad \mu \neq \nu,$$

provided we take the time coördinate imaginary, as was done by Minkowski. It is assumed that the $\gamma_{\mu\nu}$ are small, i.e. that the gravitational field is weak. In the equations the $\gamma_{\mu\nu}$ and their derivatives will occur in various powers. If the $\gamma_{\mu\nu}$ are everywhere sufficiently small compared to unity one obtains a first-approximation solution of the equations by neglecting in (1) the higher powers of the $\gamma_{\mu\nu}$ (and their derivatives) compared with the lower ones. If one introduces further the $\bar{\gamma}_{\mu\nu}$ instead of the $\gamma_{\mu\nu}$ by the relations

$$\bar{\gamma}_{\mu\nu} = \gamma_{\mu\nu} - \tfrac{1}{2}\delta_{\mu\nu}\gamma_{\alpha\alpha},$$

then (1) assumes the form

$$\bar{\gamma}_{\mu\nu,\,\alpha\alpha} - \bar{\gamma}_{\mu\nu,\,\alpha\nu} - \bar{\gamma}_{\nu\alpha,\,\alpha\mu} + \bar{\gamma}_{\alpha\alpha,\,\mu\nu} = -\,2T_{\mu\nu}. \tag{3}$$

The specialization contained in (2) is conserved if one performs an infinitesimal transformation on the coördinates:

$$x_\mu' = x_\mu + \xi^\mu, \tag{4}$$

where the ξ^μ are infinitely small but otherwise arbitrary functions. One can therefore prescribe four of the $\bar{\gamma}_{\mu\nu}$ or four conditions which the $\bar{\gamma}_{\mu\nu}$ must satisfy besides the equations (3); this amounts to a specialization of the coördinate system chosen to describe the field. We choose the coördinate system in the usual way by demanding that

$$\bar{\gamma}_{\mu\alpha,\,\alpha} = 0. \tag{5}$$

It is readily verified that these four conditions are compatible with the approximate gravitational equations provided the divergence $T_{\mu\alpha,\,\alpha}$ of $T_{\mu\nu}$ vanishes, which must be assumed according to the special theory of relativity.

It turns out however that these conditions do not completely fix the coördinate system. If $\gamma_{\mu\nu}$ are solutions of (2) and (5), then the $\gamma_{\mu\nu}'$ after a transformation of the type (4)

$$\gamma_{\mu\nu}' = \gamma_{\mu\nu} + \xi^\mu{}_{,\nu} + \xi^\nu{}_{,\mu} \tag{6}$$

are also solutions, provided the ξ^μ satisfy the conditions

$$[\xi^\mu{}_{,\nu} + \xi^\nu{}_{,\mu} - \tfrac{1}{2}\delta_{\mu\nu}(\xi^\alpha{}_{,\alpha} + \xi^\alpha{}_{,\alpha})]_{,\nu} = 0,$$

or

$$\xi^\mu{}_{,\alpha\alpha} = 0. \tag{7}$$

If a γ-field can be made to vanish by the addition of terms like those in (6), i.e., by means of an infinitesimal transformation, then the gravitational field being described is only an apparent field.

With reference to (2), the gravitational equations for empty space can be written in the form

$$\left.\begin{aligned} \bar{\gamma}_{\mu\nu,\,\alpha\alpha} &= 0. \\ \gamma_{\mu\alpha,\,\alpha} &= 0. \end{aligned}\right\} \tag{8}$$

One obtains plane gravitational waves which move in the

direction of the positive x_1-axis by taking the $\bar{\gamma}_{\mu\nu}$ of the form $\varphi(x_1 + ix_4)(= \varphi(x_1 - t))$, where these $\bar{\gamma}_{\mu\nu}$ must further satisfy the conditions

$$\left.\begin{aligned} \bar{\gamma}_{11} + i\bar{\gamma}_{14} &= 0, \\ \bar{\gamma}_{41} + i\bar{\gamma}_{44} &= 0, \\ \bar{\gamma}_{21} + i\bar{\gamma}_{24} &= 0, \\ \bar{\gamma}_{31} + i\bar{\gamma}_{34} &= 0. \end{aligned}\right\} \tag{9}$$

One can accordingly subdivide the most general (progressing) plane gravitational waves into three types:

(*a*) pure longitudinal waves,

only $\bar{\gamma}_{11}$, $\bar{\gamma}_{14}$, $\bar{\gamma}_{44}$ different from zero,

(*b*) half longitudinal, half transverse waves,

only $\bar{\gamma}_{21}$ and $\bar{\gamma}_{24}$, or only $\bar{\gamma}_{31}$ and $\bar{\gamma}_{34}$ different from zero,

(*c*) pure transverse waves,

only $\bar{\gamma}_{22}$, $\bar{\gamma}_{23}$, $\bar{\gamma}_{33}$ are different from zero.

On the basis of the previous remarks it can next be shown that every wave of type (*a*) or of type (*b*) is an apparent field, that is, it can be obtained by an infinitesimal transformation from the euclidean field ($\bar{\gamma}_{\mu\nu} = \gamma_{\mu\nu} = 0$).

We carry out the proof in the example of a wave of type (*a*). According to (9) one must set, if φ is a suitable function of the argument $x_1 + ix_4$,

$$\bar{\gamma}_{11} = \varphi, \qquad \bar{\gamma}_{14} = i\varphi, \qquad \bar{\gamma}_{44} = -\varphi,$$

hence also

$$\gamma_{11} = \varphi, \qquad \gamma_{14} = i\varphi, \qquad \gamma_{44} = -\varphi.$$

If one now chooses ξ' and ξ^4 (with $\xi^2 = \xi^3 = 0$) so that

$$\xi^1 = \chi(x_1 + ix_4), \qquad \xi^4 = i\chi(x_1 + ix_4),$$

then one has

$$\xi^1{}_{,1} + \xi'{}_{,1} = 2\chi', \qquad \xi^1{}_{,4} + \xi^4{}_{,1} = 2i\chi', \qquad \xi^4{}_{,4} + \xi^4{}_{,4} = -2\chi'.$$

These agree with the values given above for γ_{11}, γ_{14}, γ_{44} if one chooses $\chi' = \frac{1}{2}\varphi$. Hence it is shown that these waves are

apparent. An analogous proof can be carried out for the waves of type (*b*).

Furthermore we wish to show that also type (*c*) contains apparent fields, namely, those in which $\bar{\gamma}_{22} = \bar{\gamma}_{33} \neq 0$, $\bar{\gamma}_{23} = 0$. The corresponding $\gamma_{\mu\nu}$ are $\gamma_{11} = \gamma_{44} \neq 0$, all others vanishing. Such a wave can be obtained by taking $\xi' = \chi$, $\xi^4 = -i\chi$, i.e. by an infinitesimal transformation from the euclidean space. Accordingly there remain as real waves only the two pure transverse types, the non-vanishing components of which are

$$\gamma_{22} = -\gamma_{33}, \tag{c_1}$$

or

$$\gamma_{23}. \tag{c_2}$$

It follows however from the transformation law for tensors that these two types can be transformed into each other by a spatial rotation of the coördinate system about the x_1-axis through the angle $\pi/4$. They represent merely the decomposition into components of the pure transverse wave (the only one which has a real significance). Type c_1 is characterized by the fact that its components do not change under the transformations

$$x_2' = -x_2, \qquad x_1' = x_1, \qquad x_3' = x_3, \qquad x_4' = x_4,$$

or

$$x_3' = -x_3, \qquad x_1' = x_1, \qquad x_2' = x_2, \qquad x_4' = x_4,$$

in contrast to c_2, i.e. c_1 is symmetrical with respect to the x_1–x_2-plane and the x_1–x_3-plane.

We now investigate the generation of waves, as it follows from the approximate (linearized) gravitational equations, The system of the equations to be integrated is

$$\left.\begin{aligned} \bar{\gamma}_{\mu\nu,\,\alpha\alpha} &= -2T_{\mu\nu}, \\ \bar{\gamma}_{\mu\alpha,\,\alpha} &= 0. \end{aligned}\right\} \tag{10}$$

Let us suppose that a physical system described by $T_{\mu\nu}$ is found in the neighborhood of the origin of coördinates. The γ-field is then determined mathematically in a similar way to that in which an electromagnetic field is determined through an electrical current system. The usual solution is the one

given by retarded potentials

$$\bar{\gamma}_{\mu\nu} = \frac{1}{2\pi}\int \frac{[T_{\mu\nu}]_{(t-r)}}{r}\,dv. \tag{11}$$

Here r signifies the spatial distance of the point in question from a volume-element, $t = x_4/i$, the time in question.

If one considers the material system as being in a volume having dimensions small compared to r_0, the distance of our point from the origin, and also small compared to the wave-lengths of the radiation produced, then r can be replaced by r_0, and one obtains

$$\bar{\gamma}_{\mu\nu} = \frac{1}{2\pi r_0}\int [T_{\mu\nu}]_{(t-r_0)}dv,$$

or

$$\bar{\gamma}_{\mu\nu} = \frac{1}{2\pi r_0}[\int T_{\mu\nu}dv]_{(t-r_0)}. \tag{12}$$

The $\bar{\gamma}_{\mu\nu}$ are more and more closely approximated by a plane wave the greater one takes r_0. If one chooses the point in question in the neighborhood of the x_1-axis, the wave normal is parallel to the x_1 direction and only the components $\bar{\gamma}_{22}$, $\bar{\gamma}_{23}$, $\bar{\gamma}_{33}$ correspond to an actual gravitational wave according to the preceding. The corresponding integrals (12) for a system producing the wave and consisting of masses in motion relative to one another have directly no simple significance. We notice however that T_{44} denotes the (negatively taken) energy density which in the case of slow motion is practically equal to the mass density in the sense of ordinary mechanics. As will be shown, the above integrals can be expressed through this quantity. This can be done because of the existence of the energy-momentum equations of the physical system:

$$T_{\mu\alpha,\,\alpha} = 0. \tag{13}$$

If one multiplies the second of these with x_2 and the fourth with $\frac{1}{2}x_2^2$ and integrates over the whole system, one obtains two integral relations, which on being combined yield

$$\int T_{22}dv = \frac{1}{2}\frac{\partial^2}{\partial x_4^2}\int x_2^2T_{44}dv. \tag{13a}$$

Analogously one obtains

$$\int T_{33}dv = \frac{1}{2}\frac{\partial^2}{\partial x_4^2}\int x_3^2 T_{44}dv,$$

$$\int T_{23}dv = \frac{1}{2}\frac{\partial^2}{\partial x_4^2}\int x_2 x_3 T_{44}dv.$$

One sees from this that the time-derivatives of the moments of inertia determine the emission of the gravitational waves, provided the whole method of application of the approximation-equations is really justified. In particular one also sees that the case of waves symmetrical with respect to the x_1–x_2 and x_1–x_3 planes could be realized by means of elastic oscillations of a material system which has the same symmetry properties. For example, one might have two equal masses which are joined by an elastic spring and oscillate toward each other in a direction parallel to the x_3-axis.

From consideration of energy relationship it has been concluded that such a system, in sending out gravitational waves, must send out energy which reacts by damping the motion. Nevertheless, one can think of the case of vibration free from damping if one imagines that, besides the waves emitted by the system, there is present a second concentric wave-field which is propagated inward and brings to the system as much energy as the outgoing waves remove. This leads to an undamped mechanical process which is imbedded in a system of standing waves.

Mathematically this is connected with the following considerations, clearly pointed out in past years by Ritz and Tetrode. The integration of the wave-equation

$$\Box\varphi = -4\pi\rho$$

by the *retarded* potential

$$\varphi = \int \frac{[\rho]_{(t-r)}}{r}dv$$

is mathematically not the only possibility. One can also do it with

$$\varphi = \int \frac{[\rho]_{(t+r)}}{r}dv,$$

i.e. by means of the "advanced" potential, or by a mixture of the two, for example,

$$\varphi = \frac{1}{2}\int \frac{[\rho]_{(t+r)} + [\rho]_{(t-r)}}{r}\, dv.$$

The last possibility corresponds to the case without damping, in which a standing wave is present.

It is to be remarked that one can think of waves generated as described above which approximate plane waves as closely as desired. One can obtain them, for example, through a limit-process by considering the wave-source to be removed further and further from the point in question and at the same time the oscillating moment of inertia of the former increased in proportion.

II. RIGOROUS SOLUTION FOR CYLINDRICAL WAVES.

We choose the coördinates x_1, x_2 in the meridian plane in such a way that $x_1 = 0$ is the axis of rotation and x_2 runs from 0 to infinity. Let x_3 be an angle coördinate specifying the position of the meridian plane. Also, let the field be symmetrical about every plane $x_2 =$ const. and about every meridian plane. The required symmetry leads to the vanishing of all components $g_{\mu\nu}$ which contain one and only one index 2; the same holds for the index 3. In such a gravitational field only

$$g_{11}, \quad g_{22}, \quad g_{33}, \quad g_{44}, \quad g_{14}$$

can be different from zero. For convenience we now take all the coördinates real. One can further transform the coördinates x_1, x_4 so that two conditions are satisfied. As such we take

$$\left.\begin{aligned} g_{14} &= 0, \\ g_{11} &= -\, g_{44}. \end{aligned}\right\} \tag{14}$$

It can be easily shown that this can be done without introducing any singularities.

We now write

$$\left.\begin{aligned} -\, g_{11} &= g_{44} = A, \\ -\, g_{22} &= B, \\ -\, g_{33} &= C, \end{aligned}\right\} \tag{15}$$

where $A, B, C > 0$. In terms of these quantities one calculates that

$$
\left.
\begin{aligned}
2\left(R_{11} - \frac{1}{2} g_{11} R\right) &= \frac{B_{44}}{B} + \frac{C_{44}}{C} - \frac{1}{2}\left[\frac{B_4^2}{B^2} + \frac{C_4^2}{C^2}\right. \\
&\quad - \frac{B_4 C_4}{BC} + \frac{A_4}{A}\left(\frac{B_4}{B} + \frac{C_4}{C}\right) \\
&\quad \left. + \frac{B_1 C_1}{BC} + \frac{A_1}{A}\left(\frac{B_1}{B} + \frac{C_1}{C}\right)\right], \\
\frac{2A}{B}\left(R_{22} - \frac{1}{2} g_{22} R\right) &= \frac{A_{44}}{A} + \frac{C_{44}}{C} - \frac{A_{11}}{A} - \frac{C_{11}}{C} \\
&\quad + \frac{1}{2}\left[\frac{C_1^2}{C^2} - \frac{C_4^2}{C^2}\right. \\
&\quad \left. + \frac{2A_1^2}{A^2} - \frac{2A_4^2}{A^2}\right], \\
\frac{2A}{C}\left(R_{33} - \frac{1}{2} g_{33} R\right) &= \frac{A_{44}}{A} + \frac{B_{44}}{B} - \frac{A_{11}}{A} - \frac{B_{11}}{B} \\
&\quad + \frac{1}{2}\left[\frac{2A_1^2}{A^2} - \frac{2A_4^2}{A^2}\right. \\
&\quad \left. + \frac{B_1^2}{B^2} - \frac{B_4^2}{B^2}\right], \\
2\left(R_{44} - \frac{1}{2} g_{44} R\right) &= \frac{B_{11}}{B} + \frac{C_{11}}{C} - \frac{1}{2}\left[\frac{B_1^2}{B^2} + \frac{C_1^2}{C^2}\right. \\
&\quad - \frac{B_1 C_1}{BC} + \frac{A_1}{A}\left(\frac{B_1}{B} + \frac{C_1}{C}\right) \\
&\quad \left. + \frac{B_4 C_4}{BC} + \frac{A_4}{A}\left(\frac{B_4}{B} + \frac{C_4}{C}\right)\right], \\
2R_{14} &= \frac{B_{14}}{B} + \frac{C_{14}}{C} - \frac{1}{2}\left[\frac{B_1 B_4}{B^2} + \frac{C_1 C_4}{C^2}\right. \\
&\quad + \frac{A_4}{A}\left(\frac{B_1}{B} + \frac{C_1}{C}\right) \\
&\quad \left. + \frac{A_1}{A}\left(\frac{B_4}{B} + \frac{C_4}{C}\right)\right],
\end{aligned}
\right\} \qquad (16)
$$

where subscripts in the right-hand members denote differ-

entiation. If we take as field equations these expressions set equal to zero, replace the second and third by their sum and difference, and introduce as new variables

$$\left.\begin{aligned} \alpha &= \log A, \\ \beta &= \tfrac{1}{2}\log(B/C), \\ \gamma &= \tfrac{1}{2}\log(BC), \end{aligned}\right\} \qquad (15a)$$

we get

$$2\gamma_{44} + \tfrac{1}{2}[\beta_4^2 + 3\gamma_4^2 + \beta_1^2 - \gamma_1^2 - 2\alpha_1\gamma_1 - 2\alpha_4\gamma_4] = 0, \quad (17)$$

$$2(\alpha_{11} - \alpha_{44}) + 2\gamma_{11} - 2\gamma_{44} + [\beta_1^2 + \gamma_1^2 - \beta_4^2 - \gamma_4^2] = 0, \quad (18)$$

$$\beta_{11} - \beta_{44} + [\beta_1\gamma_1 - \beta_4\gamma_4] = 0, \quad (19)$$

$$2\gamma_{11} + \tfrac{1}{2}[\beta_1^2 + 3\gamma_1^2 + \beta_4^2 - \gamma_4^2 - 2\alpha_1\gamma_1 - 2\alpha_4\gamma_4] = 0, \quad (20)$$

$$2\gamma_{14} + [\beta_1\beta_4 + \gamma_1\gamma_4 - 2\alpha_1\gamma_4 - 2\alpha_4\gamma_1] = 0. \quad (21)$$

The first and fourth equations of this group give

$$\gamma_{11} - \gamma_{44} + (\gamma_1^2 - \gamma_4^2) = 0. \qquad (22)$$

The substitution

$$\gamma = \log\sigma, \qquad \sigma = (BC)^{\frac{1}{2}}, \qquad (23)$$

leads to the wave equation

$$\sigma_{11} - \sigma_{44} = 0, \qquad (24)$$

which has the solution

$$\sigma = f(x_1 + x_4) + g(x_1 - x_4), \qquad (25)$$

where f and g are arbitrary functions. Eq. (18) reduces to

$$\alpha_{11} - \alpha_{44} + \tfrac{1}{2}(\beta_1^2 - \beta_4^2 + \gamma_4^2 - \gamma_1^2) = 0. \qquad (18a)$$

Equation (17) then shows that γ cannot vanish everywhere.

We must now see whether there exist undulatory processes for which γ does not vanish. We note that such an undulatory process is represented, in the first approximation, by an undulatory β, that is by a β-function which, so far as its dependence on x_1 and also its dependence on x_4 is concerned, possesses maxima and minima; we must expect this also for a rigorous solution. We know about γ that $e^\gamma = \sigma$ satisfies the wave equation (24) and therefore takes the form (25). From this, however, the undulatory nature of this quantity

does not necessarily follow. We shall in fact show that γ can have no minima.

Such a minimum would imply that the functions f and g in (25) have minima. At a point (x_1, x_4) where this were the case we should have $\gamma_1 = \gamma_4 = 0$, $\gamma_{11} \geqq 0$, $\gamma_{44} \geqq 0$. But by (17) and (20) this is impossible. Therefore γ has no minima, that is it is not undulatory but behaves, at least in a region of space arbitrarily extended in one direction, monotonically. We shall now consider such a region of space.

It is useful to see what sort of transformations of x_1 and x_4 leave our system of equations (14) invariant. For this invariance it is necessary and sufficient that the transformation satisfy the equations

$$\left.\begin{aligned} \frac{\partial \bar{x}_1}{\partial x_1} &= \frac{\partial \bar{x}_4}{\partial x_4}, \\ \frac{\partial \bar{x}_1}{\partial x_4} &= \frac{\partial \bar{x}_4}{\partial x_1}, \end{aligned}\right\} \tag{26}$$

Thus we may arbitrarily choose $\bar{x}_1(x_1, x_4)$ to satisfy the equation

$$\frac{\partial^2 \bar{x}_1}{\partial x_1^2} - \frac{\partial^2 \bar{x}_1}{\partial x_4^2} = 0 \tag{26a}$$

and then (26) will determine the corresponding $\bar{x}_4$. Since e^γ is invariant under this transformation and also satisfies the wave equation, there exists a transformation where $\bar{x}_1$ is respectively equal or proportional to e^γ. In the *new* coördinate system we have

$$e^\gamma = a x_1$$

or

$$\gamma = \log a + \log x_1. \tag{27}$$

If we insert this expression for γ in (17)–(27) the equations reduce to the equivalent system

$$\beta_{11} - \beta_{44} + \frac{1}{x_1}\beta_1 = 0, \tag{28}$$

$$\alpha_1 = \tfrac{1}{2}x_1(\beta_1^2 + \beta_4^2) - \frac{1}{2x_1}, \tag{29}$$

and

$$\alpha_4 = x_1\beta_1\beta_4. \tag{30}$$

Equation (28) is the equation for cylindrical waves in a three-dimensional space, if x_1 denotes the distance from the axis of rotation. The equations (29) and (30) determine, for given β, the function α up to an (arbitrary) additive constant, while, by (27), γ is already determined.

In order that the waves may be regarded as waves in a euclidean space these equations must be satisfied by the euclidean space when the field is independent of x_4. This field is represented by

$$A = 1; \qquad B = 1; \qquad C = x_1^2,$$

if we denote the angle about the axis of rotation by x_3. These relations correspond to

$$\alpha = 0, \qquad \beta = -\log x_1, \qquad \gamma = \log x_1,$$

and from this we see that the equations (27)–(30) are in fact satisfied.

We have still to investigate whether *stationary* waves exist, that is waves which are purely periodic in the time.

For β it is at once clear that such solutions exist. Although it is not essential, we shall now consider the case where the variation of β with time is sinusoidal. Here β has the form

$$\beta = X_0 + X_1 \sin \omega x_4 + X_2 \cos \omega x_4,$$

where X_0, X_1, X_2 are functions of x_1 alone. From (30) it then follows that α is periodic if and only if the integral

$$\int \beta_1 \beta_4 dx_4$$

taken over a whole number of periods vanishes.

In the case of a stationary oscillation, which is represented by

$$\beta = X_0 + X_1 \sin \omega x_4,$$

this condition is actually fulfilled since

$$\int \beta_1 \beta_4 dx_4 = \int (X_0' + X_1' \sin \omega x_4) \omega X_1 \cos \omega x_4 dx_4 = 0.$$

On the other hand, in the general case, which includes the case of progressive waves, we obtain for this integral the value

$$\tfrac{1}{2}(X_1 X_2' - X_2 X_1') \omega T,$$

where T is the interval of time over which the integral is taken. This does not vanish, in general. At distances x_1 from $x_1 = 0$ great compared with the wave-lengths, a progressive wave can be represented with good approximation in a domain containing many waves by

$$\beta = X_0 + a \sin \omega(x_4 - x_1),$$

where a is a constant (which, to be sure, is a substitute for a function depending weakly on x_1). In this case $X_1 = a \cos \omega x_1$, $X_2 = - a \sin \omega x_1$, so that the integral can be (approximately) represented by $- \frac{1}{2} a \omega^2 T$, and thus cannot vanish and always has the same sign. Progressive waves therefore produce a secular change in the metric.

This is related to the fact that the waves transport energy, which is bound up with a systematic change in time of a gravitating mass localized in the axis $x = 0$.

Note.—The second part of this paper was considerably altered by me after the departure of Mr. Rosen for Russia since we had originally interpreted our formula results erroneously. I wish to thank my colleague Professor Robertson for his friendly assistance in the clarification of the original error. I thank also Mr. Hoffmann for kind assistance in translation.

A. Einstein.